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A GENERALIZATION OF A RESULT ON THE SUM OF ELEMENT ORDERS OF A FINITE GROUP

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ABSTRACT. Let G be a finite group and let $\psi(G)$ denote the sum of element orders of G. It is well-known that the maximum value of ψ on the set of groups of order n, where n is a positive integer, will occur at the cyclic group C_n. For nilpotent groups, we prove a natural generalization of this result, obtained by replacing the element orders of G with the element orders relative to a certain subgroup H of G.

1. Introduction

Let G be a finite group. In 2009, H. Amiri, S. M. Jafarian Amiri and I. M. Isaacs introduced in their paper [1] the function

$$\psi(G) = \sum_{x \in G} o(x),$$

where $o(x)$ denotes the order of x in G. They proved the following basic theorem:

Theorem 1.1. *If G is a group of order n, then $\psi(G) \le \psi(C_n)$, and we have equality if and only if G is cyclic.*

Since then many authors have studied the properties of the function $\psi(G)$ and its relations with the structure of G (see, e.g., [2–6] and [8]). We recall only that ψ is multiplicative and $\psi(C_{p^n}) = \frac{p^{2n+1}+1}{p+1}$ when p is a prime (see, e.g., of [3: Lemmas 2.2(3) and 2.9(1)]).

Given a subgroup H of G, in what follows we will consider the function

$$\psi_H(G) = \sum_{x \in G} o_H(x),$$

where $o_H(x)$ denotes the order of x relative to H, i.e., the smallest positive integer m such that $x^m \in H$. Clearly, for $H = 1$ we have $\psi_H(G) = \psi(G)$.

By replacing $\psi(G)$ with $\psi_H(G)$, we are able to generalize the above theorem for nilpotent groups.

Theorem 1.2. *Let G be a nilpotent group of order n and H be a subgroup of order m of G. Then*

$$\psi_H(G) \le \psi_{H_m}(C_n), \tag{1.1}$$

where H_m is the unique subgroup of order m of C_n.

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Note that the inequality (1.1) can easily be proved for normal subgroups H. Indeed, in this case we have

$$o_H(x) = o(xH) \text{ in } G/H, \qquad \text{for all } x \in G$$

and therefore

$$\psi_H(G) = |H|\psi(G/H) \le m\,\psi(C_{\frac{n}{m}}) = \psi_{H_m}(C_n).$$

This also shows that the equality occurs in (1.1) whenever H is normal and G/H is cyclic.

Finally, we conjecture that Theorem 1.2 is also true for non-nilpotent groups G, i.e., it is true for all finite groups G.

Most of our notation is standard and will usually not be repeated here. Elementary notions and results on groups can be found in [7].

2. Proof of the main result

Our first lemma collects two basic properties of the function $\psi_H(G)$.

Lemma 2.1. a) *If $(G_i)_{i=\overline{1,k}}$ is a family of finite groups having coprime orders and $H_i \le G_i$, $i = 1, \ldots, k$, then*

$$\psi_{H_1\times\cdots\times H_k}(G_1 \times \cdots \times G_k) = \prod_{i=1}^{k} \psi_{H_i}(G_i).$$

In particular, if G is a finite nilpotent group, $(G_i)_{i=\overline{1,k}}$ are the Sylow p_i-subgroups of G and $H = H_1 \times \cdots \times H_k \le G$, then

$$\psi_H(G) = \prod_{i=1}^{k} \psi_{H_i}(G_i).$$

b) *If G is a finite group and $H \trianglelefteq K \le G$, then*

$$\psi_H(G) \le [K : H]\,\psi_K(G) - |K| + |H|. \tag{2.1}$$

In particular, if G is a finite p-group and $H \le K \le G$ with $|H| = p^m$ and $|K| = p^{m+1}$, then

$$\psi_H(G) \le p\,\psi_K(G) - p^m(p-1). \tag{2.2}$$

P r o o f. a) Since G_i, $i = 1, \ldots, k$, are of coprime orders, for every $x = (x_1, \ldots, x_k) \in G_1 \times \cdots \times G_k$ we have

$$o_{H_1\times\cdots\times H_k}(x) = \prod_{i=1}^{k} o_{H_i}(x_i).$$

Then

$$\begin{aligned}\psi_{H_1\times\cdots\times H_k}(G_1 \times \cdots \times G_k) &= \sum_{x=(x_1,\ldots,x_k)\in G_1\times\cdots\times G_k} o_{H_1\times\cdots\times H_k}(x)\\ &= \sum_{x_1\in G_1}\cdots\sum_{x_k\in G_k} o_{H_1}(x_1)\cdots o_{H_k}(x_k)\\ &= \prod_{i=1}^{k}\Big(\sum_{x_i\in G_i} o_{H_i}(x_i)\Big) = \prod_{i=1}^{k}\psi_{H_i}(G_i),\end{aligned}$$

as desired.

b) Let $x \in G$. Then $x^{o_K(x)} \in K$ and so $x^{o_K(x)}H \in K/H$, implying that $\left(x^{o_K(x)}H\right)^{[K:H]} = H$. Thus $x^{[K:H]\, o_K(x)} \in H$, which leads to

$$o_H(x) \mid [K:H]\, o_K(x)$$

and consequently

$$o_H(x) \leq [K:H]\, o_K(x).$$

This shows that

$$\begin{aligned}\psi_H(G) = \sum_{x\in G} o_H(x) &= \sum_{x\in G\setminus H} o_H(x) + \sum_{x\in H} o_H(x)\\ &\leq [K:H] \sum_{x\in G\setminus H} o_K(x) + |H|\\ &= [K:H]\bigg(\sum_{x\in G} o_K(x) - \sum_{x\in H} o_K(x)\bigg) + |H|\\ &= [K:H]\,(\psi_K(G) - |H|) + |H|\\ &= [K:H]\,\psi_K(G) - |K| + |H|,\end{aligned}$$

completing the proof. $\square$

Remark. By taking $H = 1$ and $K \trianglelefteq G$ in (2.1), one obtains

$$\psi(G) \leq |K|\,\psi_K(G) - |K| + 1 = |K|^2\,\psi(G/K) - |K| + 1. \tag{2.3}$$

This improves the inequality in [4: Proposition 2.6]. Also, by taking $K = G$ in (2.3), we get a new upper bound for $\psi(G)$:

$$\psi(G) \leq |G|^2 - |G| + 1. \tag{2.4}$$

Note that we have equality in (2.4) if and only if G is cyclic of prime order.

Next we prove the inequality (1.1) for p-groups.

Lemma 2.2. *Let G be a p-group of order p^n and H be a subgroup of order p^m of G. Then*

$$\psi_H(G) \leq \psi_{H_{p^m}}(C_{p^n}).$$

Proof. We will proceed by induction on $[G:H]$. Obviously, the inequality holds for $[G:H] = 1$. Assume now that it holds for all subgroups of G of index less than $[G:H]$. Since every subgroup of G is subnormal, we can choose $K \leq G$ such that $H \subset K$ and $|K| = p^{m+1}$. Then, by (2.1) and the inductive hypothesis, we get

$$\begin{aligned}\psi_H(G) \leq p\,\psi_K(G) - p^m(p-1) &\leq p\,\psi_{H_{p^{m+1}}}(C_{p^n}) - p^m(p-1)\\ &= p^{m+2}\psi(C_{p^{n-m-1}}) - p^m(p-1) = p^m\left[p^2\,\frac{p^{2n-2m-1}+1}{p+1} - (p-1)\right]\\ &= p^m\,\frac{p^{2n-2m+1}+1}{p+1} = p^m\psi(C_{p^{n-m}}) = \psi_{H_{p^m}}(C_{p^n}),\end{aligned}$$

as desired. $\square$

We are now able to prove our main result.

Proof of Theorem 1.1. Let $n = p_1^{n_1} \cdots p_k^{n_k}$ be the decomposition of n as a product of prime factors. Since G is nilpotent, we have $G \cong G_1 \times \cdots \times G_k$, where $(G_i)_{i=\overline{1,k}}$ are the Sylow p_i-subgroups of G. Moreover, any subgroup H of G is of type $H \cong H_1 \times \cdots \times H_k$ with $H_i \leq G_i$, $|H_i| = p_i^{m_i}$, for all $i = 1, \dots, k$. Then, Lemmas 2.1 a), and 2.2 lead to

$$\psi_H(G) = \prod_{i=1}^{k} \psi_{H_i}(G_i) \leq \prod_{i=1}^{k} \psi_{H_{p_i^{m_i}}}(C_{p_i^{n_i}}) = \psi_{H_m}(C_n).$$

This completes the proof. □

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